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Seshadri constants on algebraic surfaces

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0. Introduction

Seshadri constants are local invariants that are naturally associated to polarized varieties. Except in the simplest cases they are very hard to control or to compute explicitly. The purpose of the present paper is to study these invariants on algebraic surfaces; we prove a number of explicit bounds on Seshadri constants and Seshadri sub-maximal curves, and we give complete results for abelian surfaces of Picard number one.

In recent years there has been considerable interest in understanding the local positivity of ample line bundles on algebraic varieties. Motivated in part by the study of linear series in connection with Fujita's conjectures, Demailly [12] captured the concept of local positivity in the *Seshadri constant*, a real number $\varepsilon(L,x)$ associated with an ample line bundle L at a point x of an algebraic variety X, which in effect measures how much of the positivity of L can be concentrated at x. Interest in Seshadri constant derives on the one hand from the fact that, via vanishing theorems, a lower bound on the Seshadri constants $\varepsilon(L,x)$ yields bounds on the number of points and jets that the adjoint series $\mathcal{O}_X(K_X+L)$ separates. On the other hand it has become increasingly clear that Seshadri constants are highly interesting invariants of algebraic varieties quite in their own right. For instance, the papers [29] and [1] address the question as to what kind of geometric information is encoded in them. We refer to Section 1 and to [13, Section 1] for more on background and motivation.

Even though on surfaces linear series are reasonably well understood thanks to powerful methods such as Reider's theorem, Seshadri constants are—as Demailly pointed out in [12]—extremely delicate already in the two-dimensional case. For instance, if X is

a generic smooth surface of degree d in $\mathbb{P}^3$, then the Seshadri constants of its hyperplane bundle are unknown when $d \geq 5$ (cf. [1] for the case $d = 4$). In light of these facts it seems interesting to study Seshadri constants in the surface case, and to aim for explicit bounds or even explicit values.

Our first result concerns surfaces that come with a fixed embedding into projective space. It is clear that in this case one has $\varepsilon(L, x) \geq 1$ at all points x, and it is natural to ask under which circumstances equality holds and which small values bigger than 1 can occur. Theorem 2.1 answers these questions.

The second result deals with line bundles that are merely assumed to be ample. Work of Ein and Lazarsfeld [14] on surfaces, and Ein-Küchle-Lazarsfeld [13] for the higher dimensional case, shows that there exist universal lower bounds on Seshadri constants if one restricts one's attention to very general points. Refinements of this type of results are due to Küchle and Steffens [19], Steffens [33] and Xu [35]. On the other hand, well-known examples due to Miranda [21, Proposition 5.12] show that there cannot exist universal lower bounds on Seshadri constants that are valid at *arbitrary* points. So any bound on Seshadri constants at not necessarily very general points needs to take into account the geometry of the polarized surface (X, L) in some way. We provide in Theorem 3.1 a bound in terms of the canonical slope of L, an invariant defined in terms of the nef cone of the surface. We also carry out a closer analysis of Miranda's examples, which illustrates the interplay between this invariant and the Seshadri constant.

We next study Seshadri sub-maximal curves at very general points, i.e. curves causing the Seshadri constant $\varepsilon(L, x)$ to be below its maximal possible value $\sqrt{L^2}$ at these points. Naturally, it is of interest to find constraints on the existence of such curves, for instance upper bounds on their L-degree. Now, as the results of Section 6 will show, the curves computing Seshadri constants can have arbitrarily high degree, and—still worse—there cannot exist a bound involving only L^2. However, working variationally as in [14] and [33], we show in Theorem 4.1 that there does exist an explicit bound for curves causing $\varepsilon(L, x)$ to be less than $\sqrt{L^2} - \delta$, for given $\delta > 0$, in terms of L^2 and δ. In a similar vein, we ask in Section 5 for the number of sub-maximal curves at a fixed (not necessarily very general) point, and we provide an explicit upper bound on this number in 5.1.

In Sections 6 to 8 we focus on the study of Seshadri constants on abelian surfaces. Our motivation to investigate this class of surfaces is twofold: First, there are up to now hardly any non-trivial examples where Seshadri constants are explicitly known. And secondly, the study of Seshadri constants on abelian varieties in general has just recently gained considerable attention (see [28], [22] and [3]). Notably, in [22] Lazarsfeld has established a surprising connection between the Seshadri constant $\varepsilon(X, L)$ of an abelian variety (X, L) and a metric invariant, the minimal period length $m(X, L)$. This allows on the one hand to get lower bounds on $\varepsilon(X, L)$ as soon as a lower bound on $m(X, L)$ is available, as is the case for principal polarizations by work of Buser and Sarnak [8] and for polarizations of arbitrary type by [3]. On the other hand, it can be used to show that certain abelian varieties (such as Jacobians [22] and Prym varieties [3]) have a period of unusually short length. For the surface case, previous joint work [4] of the author with Szemberg gave an upper bound on the Seshadri constant $\varepsilon(X, L)$ of an abelian surface

(X, L) involving the solutions of a diophantine equation. In some cases, this upper bound could be shown to be equal to $\varepsilon(X, L)$, leading to the speculation that this might always be true if (X, L) is generic. In Theorem 6.1 we will complete the picture by showing that this is in fact the case. A nice feature of this result is that it allows to explicitly compute the Seshadri constants—as well as the unique irreducible curve that accounts for it—for a whole class of surfaces. It also shows that Seshadri constants have an intriguing number-theoretic flavor in this case. In view of the papers [22] and [3] it would be most interesting to know if a similar formula exists for abelian varieties in any dimension.

In Section 7 we classify the nef cones of abelian surfaces, which we think is interesting both in view of applications to Seshadri constants and as a complement to [2]. Finally we consider Seshadri constants along finite sets in Section 8. This generalized notion, which appears implicitly already in Nagata's famous conjecture [27], has been studied previously by Xu (see [34] and [21, 5.16]) and Küchle [18]. Our purpose here is to provide a lower bound for multiple-point Seshadri constants on abelian surfaces and to study their relationship with the one-point constant.

Notation and Conventions. We work throughout over the field $\mathbb{C}$ of complex numbers. The symbol $\equiv$ denotes numerical equivalence of divisors or line bundles, whereas linear equivalence will be denoted by $\sim$. For a real number x we denote by $\lfloor x \rfloor$ is its round-down (integer part). We will say that a property holds for a *very general* point of a variety X, if it holds off the union of countably many proper closed subvarieties of X.

1. Seshadri constants

We briefly recall in this section the definition of Seshadri constants and their relationship with linear series. For a more detailed exposition we refer to [21, Section 5] and [13, Section 1].

Consider a smooth projective variety X and a nef line bundle L on X. Let

$$f : Y = \mathrm{Bl}_x(X) \longrightarrow X$$

be the blow-up of X at a point $x \in X$ and $E = f^{-1}(x)$ the exceptional divisor. The *Seshadri constant* $\varepsilon(L, x)$ is by definition the maximal real number ε such that the line bundle $f^*L - \varepsilon E$ is nef, i.e.

$$\varepsilon(L, x) =_{\mathrm{def}} \sup \{ \varepsilon \in \mathbb{R} \mid f^*L - \varepsilon E \text{ nef} \} \ .$$

It is elementary that this can be equivalently expressed on the variety X itself as the infimum

$$\varepsilon(L, x) = \inf \left\{ \frac{L \cdot C}{\mathrm{mult}_x C} \,\middle|\, C \subset X \text{ irreducible curve through } x \right\} \ .$$

Intuitively, the number $\varepsilon(L,x)$ measures how much of the positivity of L can be concentrated at the point x. The name Seshadri constant derives from the fact that by Seshadri's criterion, L is ample if and only if its *global Seshadri constant*

$$\varepsilon(L) = \inf \{ \varepsilon(L,x) \mid x \in X \}$$

is positive.

There are interesting characterizations of Seshadri constants in terms of linear series. Recall that, for an integer $s \geq 0$, a line bundle B on X (or the linear series $|B|$) is said to *separate s-jets at x*, if B admits global sections with arbitrarily prescribed s-jet at x, i.e. if the evaluation map

$$H^0(X,B) \longrightarrow H^0(X, B \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1})$$

is onto. Let $s(B,x)$ denote the largest integer s such that $|B|$ separates s-jets at x. The relationship of the separation of jets with Seshadri constants can then be summarized as follows:

Proposition 1.1. *Let X be a smooth projective variety, $x \in X$ a point, and L an ample line bundle on X.*

(a) *For every line bundle M on X, one has*

$$\varepsilon(L,x) = \limsup_{k \longrightarrow \infty} \frac{s(M+kL,x)}{k} \ .$$

(b) *If k and s are non-negative integers satisfying the inequality*

$$k > \frac{s + \dim(X)}{\varepsilon(L,x)} \ ,$$

then the adjoint linear series $|K_X + kL|$ separates s-jets at x.

When $M = \mathcal{O}_X$ then statement (a) is Theorem 6.4 in [12]; the fact that is holds for arbitrary M can be shown using arguments as in the proof of [13, Proposition 1.1(b)]. Part (b) is an application of Kawamata-Viehweg vanishing on the blow-up of X (see [13, Proposition 1.1(a)]). Note a subtle but crucial difference between (a) and (b): Whereas statement (a) tells us that $\varepsilon(L,x)$ is determined by the asymptotic behaviour of the series $|M+kL|$ for $k \gg 0$, where M is *any* given line bundle (for instance $M = \mathcal{O}_X$ or $M = K_X$), the statement in (b), which gives information about the series $|K_X + kL|$ for $k \geq k_0$ with an explicit value for k_0, does not remain true in general when K_X is replaced by an arbitrary line bundle M (e.g. by $\mathcal{O}_X$).

2. Very ample line bundles

Consider a smooth projective surface X and a very ample line bundle L on X. If C is an irreducible curve on X and $x \in C$ a point, then there is certainly a divisor $D \in |L|$ passing through x and meeting C properly. Thus the Seshadri constant $\varepsilon(L,x)$, and hence the global Seshadri constant $\varepsilon(L)$, is always at least one. This elementary argument already gives the best possible lower bound in general, since if the surface X contains a line (when embedded via the linear series $|L|$), then equality $\varepsilon(L) = 1$ is attained. It is then natural to ask whether this is the only case where $\varepsilon(L) = 1$ occurs, and what the next possible values of $\varepsilon(L)$ for a very ample line bundle are. Theorem 2.1 below answers these questions. When dealing with a very ample line bundle L on X, we will identify X with the surface in $\mathbb{P}(H^0(X,L))$ obtained by embedding X via the linear series $|L|$ and we will write

$$\varepsilon(X,x) =_{\mathrm{def}} \varepsilon(\mathcal{O}_X(1),x) = \varepsilon(L,x) \ .$$

and

$$\varepsilon(X) =_{\mathrm{def}} \varepsilon(\mathcal{O}_X(1)) = \varepsilon(L) \ .$$

We refer to these numbers simply as *the Seshadri constants of* X, tacitly suppressing the (fixed) choice of the projective embedding.

Theorem 2.1. (a) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible surface. Then $\varepsilon(X) = 1$ if and only if X contains a line.*

(b) *For $d \geq 4$ let $\mathcal{S}_{d,N}$ denote the space of smooth irreducible surfaces of degree d in $\mathbb{P}^N$ that do not contain any lines. Then*

$$\min\{\, \varepsilon(X) \mid X \in \mathcal{S}_{d,N} \,\} = \frac{d}{d-1} \ .$$

(c) *If X is a surface in $\mathcal{S}_{d,N}$ and $x \in X$ is a point such that the local Seshadri constant $\varepsilon(X,x)$ satisfies the inequalities $1 < \varepsilon(X,x) < 2$, then it is of the form*

$$\varepsilon(X,x) = \frac{a}{b} \ ,$$

where a, b are integers with $3 \leq a \leq d$ and $a/2 < b < a$.

(d) *All rational numbers a/b with $3 \leq a \leq d$ and $a/2 < b < a$ occur as local Seshadri constants of smooth irreducible surfaces in $\mathbb{P}^3$ of degree d.*

Proof. (a) The "if" part being obvious we assume $\varepsilon(X) = 1$ and show that X contains a line. Our first claim is then:

There is an irreducible curve $C \subset X$ and a point $x \in X$ such that $\deg(C) = \mathrm{mult}_x(C)$. (2.1.1)

To prove (2.1.1) we assume by way of contradiction that there is a sequence $(C_n, x_n)_{n\geq 0}$ of irreducible curves $C_n \subset X$ and points $x_n \in X$ such that

$$\varepsilon_n =_{\text{def}} \frac{\deg(C_n)}{\operatorname{mult}_{x_n}(C_n)} \longrightarrow 1 \quad , \text{ but } \varepsilon_n > 1 \text{ for all } n \geq 0.$$

We may choose the C_n such that $\varepsilon_n < 2$ for all $n \geq 0$. Since $h^0(X, \mathcal{O}_X(1)) \geq 4$, there is for every $n \geq 0$ a divisor $D_n \in |\mathcal{O}_X(1)|$ with $\operatorname{mult}_{x_n}(D_n) \geq 2$. If D_n and C_n were to meet properly, then we would have

$$\begin{aligned} \deg(C_n) &= D_n C_n \\ &\geq \operatorname{mult}_{x_n}(D_n) \cdot \operatorname{mult}_{x_n}(C_n) \\ &\geq 2 \operatorname{mult}_{x_n}(C_n) \ , \end{aligned}$$

and hence $\varepsilon_n \geq 2$, which contradicts our assumption on ε_n above. So C_n must be a component of D_n, thus we get the estimate

$$\deg(C_n) \leq D_n^2 = \deg(X) \ . \tag{2.1.2}$$

We may assume that the curves C_n are chosen in such a way that $\varepsilon_n \leq 1 + \frac{1}{n}$, hence

$$\operatorname{mult}_{x_n}(C_n) < \deg(C_n) \leq \left(1 + \frac{1}{n}\right) \operatorname{mult}_{x_n}(C_n) \ .$$

The fact that $\deg(C_n)$ is an integer then implies that $\operatorname{mult}_{x_n}(C_n) \geq n$. So we get $\deg(C_n) > n$, which contradicts the upper bound (2.1.2) on the degree of the C_n. This establishes the claim (2.1.1).

To prove the statement in the theorem, we are now going to show that the curve C in (2.1.1) is in fact a line. To this end, consider the projection

$$\pi : \mathbb{P}^N - \{x\} \longrightarrow \mathbb{P}^{N-1}$$

from x onto a hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$. Let $H \subset \mathbb{P}^N$ be a hyperplane passing through x and not containing C. Then the equality $\deg(C) = \operatorname{mult}_x(C)$ implies that the intersection $H \cap C$ is supported entirely on the point x. So we find that the image $\pi(C - \{x\})$ does not meet the generic hyperplane H. But then $\pi(C - \{x\})$ must be finite, and hence C is a line.

(b) We first show that $\varepsilon(X) \geq \frac{d}{d-1}$ for all $X \in \mathcal{S}_{d,N}$. So let $X \in \mathcal{S}_{d,N}$. We may assume $\varepsilon(X) < 2$. Then there is an irreducible curve $C \subset X$ and a point $x \in X$ such that

$$\frac{L \cdot C}{\operatorname{mult}_x C} < 2 \tag{2.1.3}$$

Let $H \subset \mathbb{P}^N$ be a hyperplane containing the tangent plane $T_x X$. Then (2.1.3) implies that C is a component of the hyperplane section $X \cap H \in |L|$. Since this holds for all

hyperplanes H containing T_xX, we conclude that $C \subset X \cap T_xX$, i.e. C lies in the tangent plane at x. But then we know that

$$\operatorname{mult}_x C \le L \cdot C - 1 \ ,$$

since C cannot be a line, and therefore

$$\frac{L \cdot C}{\operatorname{mult}_x C} \ge \frac{L \cdot C}{L \cdot C - 1} \ge \frac{d}{d-1} \ .$$

This shows that $\varepsilon(X) \ge d/(d-1)$, as claimed. The fact that the minimum is taken on by some surface is a special case of statement (d) which we will prove below.

(c) Suppose $C \subset X$ is an irreducible curve and $x \in X$ a point such that

$$1 < \frac{L \cdot C}{\operatorname{mult}_x C} < 2 \ .$$

It follows as in (b) that C is a component of a divisor D, whose support is contained in the tangent plane T_xX, so that we have

$$1 < \deg C \le d \quad \text{and} \quad \frac{\deg C}{2} < \operatorname{mult}_x C < \deg C \ .$$

The intersection $X \cap T_xX$ consists of only finitely many irreducible curves, and the Seshadri constant $\varepsilon(X, x)$ is computed by one of these curves. This shows that $\varepsilon(X, x)$ is of the form a/b as claimed.

(d) It remains to show that all rational numbers a/b, where a and b satisfy the conditions specified in the theorem, occur in this way. To this end, given a and b, we choose according to Lemma 2.2(a) below an irreducible curve $C_0 \subset \mathbb{P}^2$ of degree a with a point x of multiplicity b. Further, we take a smooth curve $C_1 \subset \mathbb{P}^2$ of degree $d - a$ not passing through x. By statement (b) of Lemma 2.2, there is a smooth surface $X \subset \mathbb{P}^3$ such that the divisor $C_0 + C_1$ is a hyperplane section of X. According to the arguments in the proof of (b), the curve C computing $\varepsilon(X, x)$ is a component of the intersection $X \cap T_xX$, and therefore $C = C_0$. So we conclude

$$\varepsilon(X, x) = \frac{L \cdot C_0}{\operatorname{mult}_x C_0} = \frac{a}{b} \ .$$

This completes the proof of the theorem. $\square$

Lemma 2.2. (a) *Let $p \in \mathbb{P}^2$ be a point and let m and d be positive integers with $m < d$. Then there are irreducible curves of degree d with a point of multiplicity m at p.*

(b) *For every reduced divisor $D \subset \mathbb{P}^2$ there is a smooth surface $X \subset \mathbb{P}^3$ such that D is a hyperplane section of X.*

Note that in (b) the assumption that D be reduced is essential: A non-reduced divisor can never be a hyperplane section of a smooth surface in projective three-space, since the Gauß map of a smooth hypersurface of degree ≥ 2 is finite.

Proof. (a) Fix a point $p \in \mathbb{P}^2$ and consider on the blow-up $f : X \longrightarrow \mathbb{P}^2$ at p with exceptional divisor E over p the line bundle

$$M_{d,m} =_{\text{def}} f^*\mathcal{O}_{\mathbb{P}^2}(d) - mE \ .$$

Since $\mathcal{O}_{\mathbb{P}^2}(d)$ is d-jet ample, the line bundle $M_{d,m}$ is globally generated whenever $m \leq d$ (cf. [5, Lemma 3.1]). Moreover, for $m < d$ we have $M_{d,m}^2 > 0$, so that the linear series $|M_{d,m}|$ is not composed with a pencil. Now take an irreducible element $C_0 \in |M_{d,m}|$. Its direct image $C = f_*C_0$ satisfies our requirements.

(b) Let d denote the degree of D. We may certainly assume $d \geq 2$. Since D is reduced, we can choose a curve $D' \subset \mathbb{P}^2$ of degree $d-1$ meeting D transversely. Let g and g' be affine equations of D and D' respectively, and consider the surface $X \subset \mathbb{P}^3$ with the affine equation

$$f(x_1, x_2, x_3) = g(x_1, x_2) + x_3 \cdot g'(x_1, x_2) \ .$$

It contains the divisor D as its intersection with the plane $H = \{x_3 = 0\}$, so it remains to show that X is smooth. Suppose then to the contrary that there is a singularity $p = (p_1, p_2, p_3)$ of X. Looking at the derivative of f with respect to the coordinate x_3 we obtain $g'(p_1, p_2) = 0$, so that the equation $f(p) = 0$ implies $g(p_1, p_2) = 0$. This means that the point $p' = (p_1, p_2)$ is a point of intersection of the divisors D and D'. Therefore the 1-jet $j_p^1(f)$ of f at p is given by

$$0 = j_p^1(f) = j_{p'}^1(g) + j_p^1(x_3 \cdot g') = j_{p'}^1(g) + p_3 \cdot j_{p'}^1(g') \ .$$

But this says that the divisors D and D' have a common tangent at p', contradicting the choice of D'. $\square$

Remark 2.3. Consider a smooth surface $X \subset \mathbb{P}^3$ of degree ≥ 3. By Theorem 2.1(a), the locus

$$\{\, x \in X \mid \varepsilon(X, x) = 1 \,\} \tag{2.3.1}$$

is the union of all lines on X. While this locus is empty for generic X, special surfaces often contain quite a number of lines. In any event, there are always only finitely many lines on a *smooth* surface, and it is an interesting, yet unsolved, classical problem to determine the *maximal* number $\ell(d)$ of lines that can lie on a smooth surface of degree d in $\mathbb{P}^3$ for any given $d \geq 3$. The only numbers that are explicitly known are $\ell(4) = 64$ (see [31]), and of course $\ell(3) = 27$. For $d \geq 5$, the best general bounds available at present are

$$3d^2 \leq \ell(d) \leq 11d^2 - 28d - 8$$

(see [10, Sect. 5.1] for the lower bound and [31, Sect. 4] for the upper bound, cf. also [24, Sect. 2.4 and 5]). It would be interesting to know if the characterization (2.3.1) could be used to derive upper bounds on $\ell(d)$. More generally, one may ask if there are explicit bounds on the degree of the loci

$$\{ x \in X \mid \varepsilon(X, x) \leq a \} \qquad \text{for } a \geq 1 \ .$$

3. Bounds on global Seshadri constants

In Section 2 we dealt with the Seshadri constants of very ample line bundles. For these line bundles the lower bound $\varepsilon(L) \geq 1$ is obvious. If we consider ample bundles, however, the situation changes quite dramatically: One knows from Miranda's examples (see [21, Proposition 5.12]) that the global Seshadri constant $\varepsilon(L)$ can become arbitrarily small. More precisely, there are sequences of polarized varieties (X_k, L_k) such that $\varepsilon(L_k) < 1/k$. So any lower bound on $\varepsilon(L)$ has to involve the geometry of the polarized variety (X, L) in some way. A closer analysis of Miranda's examples (see the proof of Proposition 3.3 below) shows that the positivity of the line bundles L_k with respect to the canonical divisor K_X get smaller and smaller as k grows. Theorem 3.1 will show that this behaviour is necessary for the fact that the Seshadri constant gets small. The number that accounts for the relation of L and K_X in this context is the slope of L relative to K_X with respect to the nef cone $\mathrm{Nef}(X)$ in the vector space $N^1(X)$ of real-valued classes of codimension one on X, i.e. the minimum

$$\sigma(L) =_{\mathrm{def}} \min \{ s \in \mathbb{R} \mid \mathcal{O}_X(sL - K_X) \text{ is nef} \} \ .$$

We will simply refer to the number $\sigma(L)$ as the *canonical slope* of L. The intuition here is that a line bundle L with large canonical slope is 'bad' in the sense that one needs a high multiple of L to reach the positivity of K_X.

Using this notion, our result can be stated as follows:

Theorem 3.1. *Let X be a smooth projective surface and L an ample line bundle on X. Then the global Seshadri constant of L is bounded in terms of the canonical slope of L by*

$$\varepsilon(L) \geq \frac{2}{1 + \sqrt{4\sigma(L) + 13}} \ .$$

Remarks 3.2. (a) For $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ one has $\sigma(L) = -3$ and $\varepsilon(X) = 1$, so that equality holds in Theorem 3.1.

(b) As for another example, consider a K3 surface X. For any ample line bundle L on X one has $\sigma(L) = 0$, hence the theorem gives

$$\varepsilon(L) \geq \frac{2}{1 + \sqrt{13}} = 0.434 \ldots$$

It is easy to see that the optimal statement in this situation is $\varepsilon(L) \geq 1/2$.

(c) Suppose that X is a surface of general type with K_X ample. We have $\sigma(K_X) = 1$, and hence

$$\varepsilon(K_X) \geq \frac{2}{1+\sqrt{17}} = 0.390...$$

Of course, we do not expect that this particular bound is sharp. It would however be interesting to know how far it is from being optimal.

(d) On any smooth projective surface one has the lower bound $\sigma(L) \geq -3$ (for instance by Mori's theorem or by Reider's criterion for global generation of line bundles). On the other hand, the invariant $\sigma(L)$ can become arbitrarily large, as already smooth surfaces in $\mathbb{P}^3$ show (see (e)).

(e) One certainly cannot expect that $\sigma(L)$ alone fully accounts for the behaviour of the Seshadri constant. Consider for instance a smooth surface $X \subset \mathbb{P}^3$ of degree d and take $L = \mathcal{O}_X(1)$. Then one always has $\sigma(L) = d - 4$, whereas the value of $\varepsilon(L)$ depends on the geometry of X (see e.g.[1]).

Proof of Theorem 3.1. Let $C \subset X$ be an irreducible curve, $x \in X$ a point, and $m = \operatorname{mult}_x C$. The idea is to use that fact that a point of multiplicity m causes the geometric genus of a curve to drop by at least $\binom{m}{2}$, in order to derive an upper bound on m. Specifically, we have by the adjunction formula

$$1 + \frac{1}{2}C(C+K_X) = p_a(C) \geq p_a(C) - p_g(C) \geq \binom{m}{2} ,$$

so that we get

$$m(m-1) \leq 2 + C(C+K_X) .$$

Now, by assumption, $\mathcal{O}_X(\sigma(L)L - K_X)$ is nef, so that in particular $K_X \cdot C \leq \sigma(L)L \cdot C$, and therefore

$$C(C+K_X) \leq C^2 + \sigma(L)L \cdot C .$$

This gives an upper bound on the multiplicity m of C at x:

$$m(m-1) \leq C^2 + \sigma(L)L \cdot C + 2 ,$$

which, upon using the Hodge index theorem, implies

$$m \leq \frac{1}{2} + \sqrt{\frac{(L \cdot C)^2}{L^2} + \sigma(L)L \cdot C + \frac{9}{4}} .$$

The upshot of these considerations is that

$$\varepsilon(L) \geq \min_{d \geq 1} \frac{d}{\frac{1}{2} + \sqrt{\frac{d^2}{L^2} + \sigma(L)d + \frac{9}{4}}} .$$

One checks now, for instance using a little elementary analysis, that this minimum is taken on at $d = 1$. $\square$

In order to shed some more light on the interplay of the invariants $\varepsilon(L)$ and $\sigma(L)$, we will now have a closer look at Miranda's examples. In fact, we will give a generalized version of [21, Proposition 5.12], which shows that the occurence of small Seshadri constants is not as exceptional as it might appear judging from Miranda's examples, which are originally constructed as certain blow-ups of $\mathbb{P}^2$. The following proposition shows that actually a suitable blow-up of *any* surface with Picard number one contains ample line bundles with Seshadri constants below any prescribed bound.

Proposition 3.3. *Let X be a smooth projective surface of Picard number one. Then:*

(a) *For every integer $r > 0$ there are line bundles L_k, $k \geq 1$, on suitable blow-ups Y_k of X such that $L_k^2 \geq r^2$, $L_k \cdot C \geq r$ for all curves C on Y, but*

$$\varepsilon(L_k) \leq \frac{1}{k} .$$

(b) *For $X = \mathbb{P}^2$ and $r = 1$ the bundles L_k can be chosen in such a way that for every real number $\delta > 0$ we have*

$$\varepsilon(L_k) - \varepsilon_0(L_k) < \frac{\delta}{k} \qquad \text{for } k \gg 0 ,$$

where $\varepsilon_0(L_k)$ denotes the lower bound (in terms of the canonical slope $\sigma(L_k)$) given by Theorem 3.1.

Remark 3.4. Note in particular that, while $\varepsilon(L_k)$ gets arbitrarily small, the stated intersection properties of L_k allow one to achieve that the adjoint linear series $|K_X + L_k|$ generates an arbitrarily prescribed number of jets at any point, and hence has an arbitrarily large global Seshadri constant $\varepsilon(K_X + L_k)$. More concretely, given any integer $s > 0$, it suffices to choose $r \geq s^2 + 4s + 5$ in order to force the linear series $|K_X + L_k|$ to generate s-jets at any point of X (e.g. by [21, Corollary 7.5]). But this implies by Proposition 1.1 that $\varepsilon(K_X + L_k) \geq s$. So, from the point of view of Seshadri constants, the linear series $|L_k|$ itself does not benefit from the numerical positivity of L_k (expressed in terms of its self-intersection and its intersection with curves), whereas the adjoint linear series $|K_X + L_k|$ does.

In order to prove the proposition, we will need the following lemma:

Lemma 3.5. *Let L be an ample line bundle whose class generates the Néron-Severi group of a smooth surface X. Consider for $d > 0$ the space $\mathcal{R}_d$ of reducible divisors in the linear series $|\mathcal{O}_X(dL)|$. Then there is a constant c such that*

$$\operatorname{codim}(\mathcal{R}_d, |\mathcal{O}_X(dL)|) \geq dL^2 + c .$$

Proof of Lemma 3.5. By Serre vanishing, there is an integer n_0 such that for $n \geq n_0$ one has

$$H^i(X, \mathcal{O}_X(nL)) = 0 \quad \text{for } i > 0 \; . \tag{3.5.1}$$

Moreover, the integer n_0 can be chosen in such a way that the vanishing remains true even if L is replaced by a numerically equivalent line bundle (cf. [15, Theorem 5.1]). To prove the lemma, it is enough to show that for $1 \leq a < d$ and for every $P \in \mathrm{Pic}^0(X)$

$$\operatorname{codim}\left(|\mathcal{O}_X(aL) \otimes P| + |\mathcal{O}_X((d-a)L) \otimes P^{-1}|, |\mathcal{O}_X(dL)|\right) \geq dL^2 + c \tag{3.5.2}$$

for some constant c independent of d and P, and it suffices to consider $d \geq 2n_0$. We distinguish between two cases. Suppose first that $a \geq n_0$ and $d - a \geq n_0$. Thanks to (3.5.1) we have then by Riemann-Roch

$$\begin{aligned}
&h^0(X, \mathcal{O}_X(dL)) - h^0(X, \mathcal{O}_X(aL) \otimes P) - h^0(X, \mathcal{O}_X((d-a)L) \otimes P^{-1}) \\
&\qquad = \chi(X, \mathcal{O}_X(dL)) - \chi(X, \mathcal{O}_X(aL) \otimes P) - \chi(X, \mathcal{O}_X((d-a)L) \otimes P^{-1}) \\
&\qquad = a(d-a)L^2 - \chi(O_X) \\
&\qquad \geq (d-1)L^2 - \chi(\mathcal{O}_X) \\
&\qquad = dL^2 + \text{const} \; ,
\end{aligned}$$

which proves the assertion (3.5.2) in this case. In the alternative case, we may by symmetry assume that $a < n_0$ and $d - a > n_0$. Then we have

$$h^0(X, \mathcal{O}_X((d-a)L) \otimes P^{-1}) \leq h^0(X, \mathcal{O}_X((d-1)L) \otimes P^{-1}) \; , \tag{3.5.3}$$

and, using the abbreviation

$$b =_{\text{def}} \max \left\{ h^0(X, \mathcal{O}_X(kL) \otimes P) \,\middle|\, 1 \leq k \leq n_0, \; P \in \mathrm{Pic}^0(X) \right\} \; ,$$

we obtain upon using (3.5.1) and (3.5.3) the estimate

$$\begin{aligned}
&h^0(X, \mathcal{O}_X(dL)) - h^0(X, \mathcal{O}_X(aL) \otimes P) - h^0(X, \mathcal{O}_X((d-a)L) \otimes P^{-1}) \\
&\qquad\qquad \geq h^0(X, \mathcal{O}_X(dL)) - b - h^0(X, \mathcal{O}_X((d-1)L) \otimes P^{-1}) \\
&\qquad\qquad = \chi(X, \mathcal{O}_X(dL)) - b - \chi(X, \mathcal{O}_X((d-1)L) \otimes P^{-1}) \\
&\qquad\qquad = \tfrac{1}{2}(2d-1)L^2 - b - \tfrac{1}{2}L \cdot K_X \\
&\qquad\qquad = dL^2 + \text{const} \; ,
\end{aligned}$$

and this completes the proof of the lemma. □

Proof of Proposition 3.3. (a) We will follow Miranda's construction to exhibit line bundles with the properties asserted in the proposition. We start by choosing an ample generator H of $NS(X)$, fixing an integer $k \geq 1$ and setting $m = rk$. For sufficiently large $d > 0$, the line bundle $\mathcal{O}_X(dH)$ will be $(m+1)$-jet ample, so that we can find an irreducible curve

$D \in |dH|$ with a point x of multiplicity $\geq m$. In view of Lemma 3.5 we can arrange, by possibly increasing d, that there is a pencil P of irreducible curves in $|dH|$ containing D. We may further assume that P has d^2H^2 distinct base points $p_1, \ldots, p_{d^2H^2} \in X$. Consider the blow-up $f : Y = Y_k \longrightarrow X$ of X at these points and the induced pencil

$$\widehat{P} = f^*P - \sum_{i=1}^{d^2H^2} E_i \ ,$$

where $E_i = f^{-1}(p_i)$. The proper transform $\widehat{D}$ of D has multiplicity $\geq m$ at the point $f^{-1}(x)$. Fix now an integer $a \geq 2$ and consider the line bundle

$$L = L_k =_{\text{def}} \mathcal{O}_Y(r(a\widehat{D} + E_1)) \ .$$

For the intersection numbers of L we have the bounds

$$L^2 = r^2(2a-1) \geq r^2 \tag{3.5.4}$$

and

$$\begin{array}{rcl} L \cdot \widehat{D} & = & r(a\widehat{D}^2 + E_1 \cdot \widehat{D}) = r > 0 \\ L \cdot E_1 & = & r(a\widehat{D} \cdot E_1 + E_1^2) = r(a-1) > 0 \ . \end{array} \tag{3.5.5}$$

The map $Y \longrightarrow \mathbb{P}^1$ induced by $\widehat{P}$ is a fibration with irreducible fibres and with section E_1. Therefore, by the Nakai-Moishezon criterion, the inequalities (3.5.4) and (3.5.5) imply that L is ample. Due to the existence of the singular curve $\widehat{D}$, its Seshadri constant is bounded from above by

$$\varepsilon(L) \leq \varepsilon(L, f^{-1}(x)) \leq \frac{L \cdot \widehat{D}}{\operatorname{mult}_{f^{-1}(x)} \widehat{D}} \leq \frac{r}{m} = \frac{1}{k} \ ,$$

whereas of course $L \cdot C \geq r$ for all curves C on Y, since L is an r-th power.

(b) Note first that for $X = \mathbb{P}^2$, $r = 1$ and $d \gg 0$ we may take $d = m + 1 = k + 1$. We now determine an upper bound on the canonical slope of the bundles L_k for $k \gg 0$. Writing $E = \sum_{i=1}^{d^2} E_i$, we have

$$K_Y = f^*K_X + E = -3f^*H + E \ ,$$

so that we find

$$K_Y \cdot \widehat{D} = d(d-3), \ K_Y \cdot E_1 = -1, \ K_Y \cdot L_k = ad^2 - 3ad - 1,$$

and $K_Y^2 = 9 - d^2$. The line bundle $sL_k - K_Y$ therefore satisfies

$$\begin{array}{rcl} (sL_k - K_Y)\widehat{D} & = & s - d(d-3) \\ (sL_k - K_Y)E_1 & = & s(a-1) + 1 \end{array}$$

and

$$(sL_k - K_Y)^2 = (2a-1)s^2 - 2s(ad^2 - 3ad - 1) + 9 - d^2 \ .$$

Fix now a real number $\eta > 1$. One checks then that for $d \gg 0$ the Nakai-Moishezon criterion implies that $sL_k - K_Y$ is ample for $s \geq \eta d^2$, and hence

$$\sigma(L_k) < \eta(k+1)^2 \quad \text{for } k \gg 0 \ .$$

We therefore get the estimate

$$\varepsilon(L_k) - \varepsilon_0(L_k) \leq \frac{1}{k} - \frac{2}{1 + \sqrt{4\eta(k+1)^2 + 13}} \ ,$$

and for $k \gg 0$ the latter expression gets smaller than δ/k, if η is chosen sufficiently close to 1. This completes the proof of the proposition. □

4. The degree of sub-maximal curves

In this section we show how the techniques from [14] can be used to derive an explicit bound on the degrees of the irreducible curves leading to sub-maximal Seshadri constants at very general points. Specifically, suppose that C is an irreducible curve and $x \in X$ a point such that the quotient

$$\varepsilon_{C,x} =_{\text{def}} \frac{L \cdot C}{\operatorname{mult}_x C}$$

is less than $\sqrt{L^2}$. In other words, the curve C causes the Seshadri constant $\varepsilon(L, x)$ to be at most $\varepsilon_{C,x} < \sqrt{L^2}$. We will briefly refer to curves with this property as *Seshadri sub-maximal curves*. From the point of view of Seshadri constants, these are the most interesting curves on X, because they account for the failure of L to have maximal positivity. It is therefore highly desirable to obtain as much information about them as possible. The following result provides an upper bound on the degree of a Seshadri sub-maximal curve C in terms of $\varepsilon_{C,x}$.

Theorem 4.1. *Let X be a smooth projective surface and let L be an ample line bundle on X. Further, let $x \in X$ be a very general point and $C \subset X$ an irreducible curve passing through x such that*

$$\varepsilon_{C,x} < \sqrt{L^2} \ .$$

Then the degree of C with respect to L is bounded as follows:

$$L \cdot C < \frac{L^2}{\sqrt{L^2} - \varepsilon_{C,x}} \ .$$

So, roughly speaking, the theorem says that only curves of small degree can force $\varepsilon(L,x)$ to be small at very general points.

Remark 4.2. It is also useful to think of the theorem as giving an upper bound on the self-intersection of C. In fact, combining the inequality in the theorem with the Hodge index theorem yields the bound

$$C^2 < \frac{L^2}{(\sqrt{L^2}-\varepsilon_{C,x})^2} \ .$$

Consider for instance the case when $\varepsilon(L,x) \le \sqrt{L^2}-1$. The theorem implies then the existence of an irreducible curve C passing through x such that

$$L\cdot C < L^2 \quad \text{and} \quad C^2 < L^2 \ .$$

In the proof of the theorem we will make use of the following result of Ein and Lazarsfeld in the spirit of [33].

Proposition 4.3 (Ein-Lazarsfeld [14]). *Let X be a smooth projective surface and let $(C_t)_{t\in T}$ be a non-trivial 1-parameter family of irreducible curves $C_t\subset X$. Suppose that $(x_t)_{t\in T}$ is a family of points $x_t\in C_t$ and m an integer such that*

$$\operatorname{mult}_{x_t} C_t \ge m$$

for all $t\in T$. Then

$$C_t^2 \ge m(m-1) \ .$$

Proof of Theorem 4.1. Let $m=\operatorname{mult}_x C$. Since x is very general in X, there exists a non-trivial family $(C_t)_{t\in T}$ of irreducible curves $C_t\subset X$ and a family $(x_t)_{t\in T}$ of points $x_t\in C_t$ such that $\operatorname{mult}_{x_t} C_t\ge m$ and $(C_{t_0},x_{t_0})=(C,x)$ for some $t_0\in T$. Proposition 4.3 then implies in particular

$$C^2 \ge m(m-1) \ . \tag{4.3.1}$$

Suppose now that α is a real number with

$$\frac{L\cdot C}{m} < \alpha \le \sqrt{L^2}$$

From these inequalities we obtain $\alpha L\cdot C < \alpha^2 m \le mL^2$ and hence

$$\alpha\cdot\frac{L\cdot C}{L^2} < m \ . \tag{4.3.2}$$

Now assume by way of contradiction that

$$\frac{L^2}{L\cdot C} \le \sqrt{L^2} - \frac{L\cdot C}{m} \ .$$

This implies that some multiple of the rational number $L^2/L\cdot C$ is contained in the interval $(L\cdot C/m, \sqrt{L^2}]$, say

$$\frac{L\cdot C}{m} < k\frac{L^2}{L\cdot C} \le \sqrt{L^2}$$

with a suitable integer k. Taking $\alpha = kL^2/L\cdot C$ and using the inequality (4.3.2) we then have

$$k < m \ .$$

The crucial point of the proof is now an elementary diophantine argument in the spirit of [33]: Since k is an integer, the previous inequality implies $k \le m-1$. This slight improvement on the bound suffices to establish a contradiction. In fact, combining the inequality $k \le m-1$ with the bound (4.3.1) and with the Hodge index theorem, one obtains

$$\begin{aligned} m(m-1) &\le C^2 \\ &\le \sqrt{\frac{C^2}{L^2}}\cdot L\cdot C \\ &< m\alpha\sqrt{\frac{C^2}{L^2}} \\ &= mk\frac{L^2}{L\cdot C}\sqrt{\frac{C^2}{L^2}} \\ &\le mk \\ &\le m(m-1), \end{aligned}$$

which is absurd, and this completes the proof of the theorem. □

As an application we give a quick proof in the surface case of a result by Nakamaye [28], which characterizes the abelian surfaces of Seshadri constant one.

Corollary 4.4. *Let (X, L) be a polarized abelian surface with $\varepsilon(L) = 1$. Then (X, L) is a polarized product of elliptic curves,*

$$X = E_1 \times E_2, \quad L = \mathcal{O}_X(d(E_1\times 0) + (0\times E_2)) \ ,$$

where $d = L^2/2$.

Proof. Fix a point $x \in X$. By assumption, there is for every $\delta > 0$ an irreducible curve $C \subset X$ such that

$$\frac{L\cdot C}{\operatorname{mult}_x C} < 1+\delta \ .$$

Since on abelian varieties the Seshadri constant is independent of the point, we can apply the theorem to get the inequality

$$C^2 < \frac{L^2}{\left(\sqrt{L^2} - (1+\delta)\right)^2} \tag{4.4.1}$$

If $L^2 \geq 6$, then for small δ the value of the expression on the right hand side is less than 2, so that we find $C^2 = 0$. Thus C is an elliptic curve, and hence $\operatorname{mult}_x C = 1$. This implies $L \cdot C = 1$ and the assertion follows immediately. If $L^2 \leq 4$, then the inequality (4.4.1) implies $C^2 \leq 2$. So either $C^2 = 0$, where we conclude as before, or else $C^2 = 2$. In the latter case C is a hyperelliptic curve of genus 2, and therefore again $\operatorname{mult}_x C = 1$. This implies $L \cdot C = 1$, which however is impossible by the Hodge index theorem. $\square$

5. On the number of sub-maximal curves

The canonical slope of an ample line bundle, which was used in Sect. 3 to obtain a lower bound on the global Seshadri constant, will also come into play when we consider the following enumerative question: Let L be an ample line bundle on a smooth projective surface X and a point $x \in X$. Given a real number $a > 0$, what can we say about the number $\nu(L, x, a)$ of irreducible curves such that

$$\frac{L \cdot C}{\operatorname{mult}_x C} < a \ ,$$

if it is finite at all? Of course, one cannot expect to actually determine the number $\nu(L, x, a)$ in general. However, we can give an explicit upper bound in terms of the canonical slope $\sigma(L)$ when a is a rational number $< \sqrt{L^2}$:

Proposition 5.1. *Let X be a smooth projective surface and L an ample line bundle on X. Suppose that a point $x \in X$ and a rational number $a < \sqrt{L^2}$ is given. Set*

$$\delta = (\sigma(L) \cdot L^2 + a)^2 - 8 \cdot \chi(\mathcal{O}_X)(L^2 - a^2) \ .$$

Then

$$\nu(L, x, a) \leq kL^2 \ ,$$

where $k > \sigma(L)$ is an integer such that the number $k \cdot a$ is integral and, in case $\delta \geq 0$, such that

$$k > \frac{\sigma(L) \cdot L^2 + a + \sqrt{\delta}}{2(L^2 - a)} \ .$$

The idea for the proof of the proposition lies in the following useful observation:

Lemma 5.2. *Let X be a smooth projective surface and L an ample line bundle on X. Given a real number $\xi > 0$ and a point $x \in X$, suppose that for some $k > 0$ there is a divisor $D \in |\mathcal{O}_X(kL)|$ such that*

$$\frac{L \cdot D}{\operatorname{mult}_x D} \leq \xi\sqrt{L^2}\ .$$

Then every irreducible curve $C \subset X$ satisfying the inequality

$$\frac{L \cdot C}{\operatorname{mult}_x C} < \frac{1}{\xi}\sqrt{L^2}$$

is a component of D.

For instance, this implies that an irreducible curve $C \in |\mathcal{O}_X(kL)|$ with $L \cdot C/\operatorname{mult}_x C < \sqrt{L^2}$ computes the Seshadri constant $\varepsilon(L,x)$.

Proof. Suppose to the contrary that D and C intersect properly. Then we get

$$\begin{aligned} kL \cdot C &= D \cdot C \geq \operatorname{mult}_x D \cdot \operatorname{mult}_x C \\ &> \frac{L \cdot D}{\xi\sqrt{L^2}} \cdot \frac{\xi L \cdot C}{\sqrt{L^2}} \\ &= kL \cdot C\ , \end{aligned}$$

and this is a contradiction. □

Proof of Proposition 5.1. The idea is simple: Find a divisor $D \in |\mathcal{O}_X(kL)|$ such that the quotient $L \cdot D/\operatorname{mult}_x D$ is sufficiently small; its degree will then by means of Lemma 5.2 give an upper bound for $\nu(L,x,a)$. Turning to the details, let $\sigma = \sigma(L)$ and $k > \sigma$. We then have

$$H^i(X, \mathcal{O}_X(kL)) = H^i(X, \mathcal{O}_X(K_X + (k-\sigma)L + (\sigma L - K_X))) = 0 \quad \text{for } i > 0$$

by Kodaira vanishing, since $(k-\sigma)L$ is ample and $\sigma L - K_X$ is nef. Therefore

$$\begin{aligned} h^0(X, \mathcal{O}_X(kL)) &= \chi(X, \mathcal{O}_X(kL)) \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}kL(kL - K_X) \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}kL\left((k-\sigma)L + (\sigma L - K_X)\right) \\ &\geq \chi(\mathcal{O}_X) + \frac{1}{2}k(k-\sigma)L^2\ . \end{aligned}$$

On the other hand, we have for $m > 0$

$$h^0(X, \mathcal{O}_X/\mathcal{I}_x^m) = \binom{m+1}{2}\ ,$$

so that the linear series $|\mathcal{O}_X(kL)\otimes\mathcal{I}_x^m|$ will be non-empty as soon as

$$\chi(\mathcal{O}_X)+\frac{1}{2}k(k-\sigma)L^2-\frac{1}{2}m(m+1)>0\ .$$

If we take $m=k\cdot a$, which by assumption is an integer, then this condition is equivalent to the quadratic inequality

$$k^2(L^2-a^2)-k(\sigma L^2+a)+2\chi(\mathcal{O}_X)>0\ . \tag{5.2.1}$$

So if δ, its discriminant, is negative, then $|\mathcal{O}_X(kL)\otimes\mathcal{I}_x^{ka}|\neq\emptyset$, since by assumption $a<\sqrt{L^2}$. If $\delta\geq 0$, then the linear series in question will be non-empty whenever $k>k_0$, where k_0 is the bigger root of the quadratic polynomial in (5.2.1).

In either case, taking a divisor $D\in|\mathcal{O}_X(kL)\otimes\mathcal{I}_x^{ka}|$, we have

$$\frac{L\cdot D}{\operatorname{mult}_x D}=\frac{kL^2}{\operatorname{mult}_x D}\leq\frac{kL^2}{ka}=\xi\sqrt{L^2}\ ,$$

where we set $\xi=_{\mathrm{def}}\sqrt{L^2}/a$. If now $C\subset X$ is an irreducible curve with

$$\frac{L\cdot C}{\operatorname{mult}_x C}<a=\frac{1}{\xi}\sqrt{L^2}\ ,$$

then, by Lemma 5.2, it is a component of D. This implies the assertion. $\square$

6. Seshadri constants of polarized abelian surfaces

Consider a polarized abelian variety (X,L). By homogeneity, the Seshadri constant $\varepsilon(L,x)$ is independent of the point $x\in X$, so it is an invariant of the polarized variety (X,L). We will denote it by $\varepsilon(X,L)$. There has been recent interest in the study of Seshadri constants of abelian varieties: Using symplectic blowing up in the spirit of [23], Lazarsfeld has established an interesting connection between Seshadri constants and minimal period lengths, leading in particular to a lower bound on $\varepsilon(X,L)$ for the principally polarized case. Generalizing the approach of Buser and Sarnak in [8], a lower bound on $\varepsilon(X,L)$ for arbitrary polarizations has been given in [3].

For the surface case, where one hopes for more specific results, an upper bound on $\varepsilon(X,L)$ involving the solutions of a diophantine equation was given in [4]. An interesting consequence of this result is that on abelian surfaces Seshadri constants are always rational. In certain cases the upper bound was shown to be equal to $\varepsilon(X,L)$, and it was tempting to hope that this might always be true if (X,L) is general. In Theorem 6.1 we will complete the picture by showing that this is in fact the case. A nice feature of this result is that it allows to explicitly compute the Seshadri constants for a whole class of surfaces. It also allows to determine the unique irreducible curve that computes $\varepsilon(X,L)$.

We show:

Theorem 6.1. *Let (X, L) be a polarized abelian surface of type $(1, d)$, $d \geq 1$, such that $NS(X) \cong \mathbb{Z}$.*

(a) *If $\sqrt{2d}$ is rational, then $\varepsilon(X, L) = \sqrt{2d}$.*

(b) *If $\sqrt{2d}$ is irrational, then*

$$\varepsilon(X, L) = 2d \cdot \frac{k_0}{l_0} = \sqrt{2d\left(1 - \frac{1}{\ell_0^2}\right)} = \frac{2d}{\sqrt{2d + \frac{1}{k_0^2}}} < \sqrt{2d} \ ,$$

where (k_0, ℓ_0) is the primitive solution of Pell's equation

$$\ell^2 - 2dk^2 = 1 \ .$$

There is (up to translation) a unique irreducible curve $C \subset X$ such that

$$\varepsilon(X, L) = \frac{L \cdot C}{\operatorname{mult}_x C} \quad \text{for some } x \in C \ ,$$

and we have either

$$\mathcal{O}_X(C) \equiv \mathcal{O}_X(k_0 L) \quad \text{and} \quad \operatorname{mult}_x C = \ell_0$$

or

$$\mathcal{O}_X(C) \equiv \mathcal{O}_X(2k_0 L) \quad \text{and} \quad \operatorname{mult}_x C = 2\ell_0 \ .$$

Moreover, the point x is the only singularity of the curve C.

In the proof we will apply the following useful lemma, which follows from 5.2:

Lemma 6.2. *Let X be a smooth projective surface, $x \in X$ a point, and L an ample line bundle on X. If there is a divisor $D \in |\mathcal{O}_X(kL)|$ for some $k > 0$ satisfying*

$$\frac{L \cdot D}{\operatorname{mult}_x D} < \sqrt{L^2} \ ,$$

then every irreducible curve $C \subset X$ with

$$\frac{L \cdot C}{\operatorname{mult}_x C} < \sqrt{L^2}$$

is a component of D.

Proof of Theorem 6.1. Assertion (a) follows from Steffens' result [33, Proposition 1] to the effect that on any surface of Picard number one we have the lower bound

$$\varepsilon(L, x) \geq \left\lfloor \sqrt{L^2} \right\rfloor$$

for very general $x \in X$.

As for (b): Replacing L by a suitable algebraically equivalent line bundle, we may assume to begin with that L is symmetric. It was shown in [4] that the linear series $|\mathcal{O}_X(2k_0L)|$ then contains an even symmetric divisor D such that $\operatorname{mult}_{e_1} D \geq 2\ell_0$, where e_1 is a fixed halfperiod on X. Thus, as in [4], we have the upper bound

$$\varepsilon(X,L) \leq \frac{L\cdot D}{\operatorname{mult}_{e_1} D} \leq \frac{k_0}{\ell_0}L^2 \ .$$

Suppose now by way of contradiction that there is an irreducible curve $C \subset X$ passing through e_1 such that

$$\frac{L\cdot C}{\operatorname{mult}_{e_1} C} < \frac{k_0}{\ell_0}L^2 \ . \tag{6.2.1}$$

Let $\iota : X \longrightarrow X$ denote the (-1)-involution on X. We have $\operatorname{mult}_{e_1} \iota^*C = \operatorname{mult}_{e_1} C$ and $L\cdot\iota^*C = L\cdot C$. Therefore, since C is algebraically equivalent to a multiple of L, Lemma 6.2 applies to C and shows that the curves C and ι^*C coincide, i.e. that C is symmetric. Lemma 6.2 also implies that C appears as a component of D, hence we have

$$\mathcal{O}_X(C) \equiv \mathcal{O}_X(k_1L) \quad \text{ with } k_1 \leq 2k_0 \text{ and } m_1 =_{\text{def}} \operatorname{mult}_{e_1} C \leq \operatorname{mult}_{e_1} D \ . \tag{6.2.2}$$

Consider the blow-up $f : \widetilde{X} \longrightarrow X$ of X at the sixteen halfperiods $e_1,\ldots,e_{16}$ and the projection $\pi : \widetilde{X} \longrightarrow K$ onto the smooth Kummer surface K of X. Since C is symmetric, its proper transform

$$C' = f^*C - \sum_{i=1}^{16} \operatorname{mult}_{e_i} C\cdot E_i$$

descends to an irreducible curve $\overline{C} \subset K$. We claim that

$$h^0(K, \mathcal{O}_K(\overline{C})) = 1 \ . \tag{6.2.3}$$

In fact, if the linear series $|\mathcal{O}_K(\overline{C})|$ were to contain a pencil, then this would give us a pencil of curves in $|\mathcal{O}_X(C)|$ with the same multiplicities at halfperiods as C. In particular, we would then have infinitely many irreducible curves satisfying (6.2.1), which however by Lemma 6.2 is impossible. This establishes (6.2.3).

Now, (6.2.3) implies $(\overline{C})^2 = -2$, since the exact sequence

$$0 \longrightarrow \mathcal{O}_K(-\overline{C}) \longrightarrow \mathcal{O}_K \longrightarrow \mathcal{O}_{\overline{C}} \longrightarrow 0$$

tells us that $H^i(K, \mathcal{O}_K(\overline{C})) = 0$ for $i > 0$, so that by Riemann-Roch

$$1 = h^0(K, \mathcal{O}_K(\overline{C})) = \chi(K, \mathcal{O}_K(\overline{C})) = 2 + \frac{1}{2}(\overline{C})^2 \ .$$

We conclude that, using the abbreviation $m_i = \operatorname{mult}_{e_i} C$,

$$k_1^2 \cdot 2d - \sum_{i=1}^{16} m_i^2 = C^2 - \sum_{i=1}^{16} m_i^2 = (C')^2 = (\pi^*\overline{C})^2 = 2(\overline{C})^2 = -4 \ , \tag{6.2.4}$$

so that we obtain the lower bound

$$k_1^2 \cdot 2d - m_1^2 \geq -4 \ . \tag{6.2.5}$$

On the other hand, we have the upper bound

$$k_1^2 \cdot 2d - m_1^2 < 0 \ , \tag{6.2.6}$$

since in the alternative case the inequality $k_1/m_1 \geq 1/\sqrt{2d}$ would imply

$$\frac{L \cdot C}{m_1} = \frac{k_1}{m_1} L^2 \geq \sqrt{2d} \ ,$$

a contradiction with (6.2.1).

So, by (6.2.5) and (6.2.6), there are only four possible values for the difference $k_1^2 \cdot 2d - m_1^2$. We will deal with these cases separately.

Case 1. Suppose that $k_1^2 \cdot 2d - m_1^2 = -4$. Then m_1 is necessarily an even number. From (6.2.4) we see that $m_i = 0$ for $i > 1$, so that in particular the multiplicities of C at all halfperiods are even. This implies that the symmetric line bundle $\mathcal{O}_X(C)$ is totally symmetric and is therefore algebraically equivalent to an even multiple of L. So k_1 is even as well. But then the pair $(k_1/2, m_1/2)$ is a solution of Pell's equation $\ell^2 - 2dk^2 = 1$. By the minimality of the solution (k_0, ℓ_0) we then have $k_1 \geq 2k_0$ and $m_1 \geq 2\ell_0$, and consequently by (6.2.2)

$$k_1 = 2k_0 \quad \text{and} \quad D = C \ .$$

But this of course makes (6.2.1) impossible.

Case 2. Suppose that $k_1^2 \cdot 2d - m_1^2 = -3$. In this case m_1 is an odd number and, looking at the equation modulo 4, we see that k_1 must be odd as well. The symmetric line bundle $\mathcal{O}_X(C)$ then has q odd halfperiods, where $q \in \{4, 6, 10, 12\}$ (cf. [7, Section 5]). But we see from (6.2.4) that $m_1^2 - \sum_{i=1}^{16} m_i^2 = -1$, so that C passes through only one halfperiod apart from e_1, which implies $q = 2$, a contradiction.

Case 3. Suppose that $k_1^2 \cdot 2d - m_1^2 = -2$. This is similar to the previous case: Now m_1 is even and k_1 is odd, as we see again by looking at the equation modulo 4. It follows from (6.2.4) that C passes through only two halfperiods apart from e_1, and we get the same kind of contradiction as in Case 2.

Case 4. Finally, suppose that $k_1^2 \cdot 2d - m_1^2 = -1$. In this case the pair (k_1, m_1) solves Pell's equation $\ell^2 - 2dk^2 = 1$, and the minimality of the solution (k_0, ℓ_0) implies

$$k_1 = k_0 \quad \text{and} \quad D = 2C \ ,$$

which does not allow (6.2.1).

We now show the assertions about C. First, the uniqueness of C is clear from Lemma 6.2. Further, since $\varepsilon(X,L)$ is computed by C, and since $L\cdot D/\operatorname{mult}_{e_i} D=\varepsilon(X,L)$, we must have $D=a\cdot C$ for some integer $a\geq 1$. The proof so far shows that either $D=2C$ (corresponding to Case 4) or $D=C$ (corresponding to Case 1). It remains to show that e_1 is the only singular point of C. The adjunction formula on K tells us that

$$p_a(\overline{C})=1+\frac{1}{2}(\overline{C})^2=0\ ,$$

so that in any event C is smooth outside of the sixteen halfperiods. Further, we have either

$$\sum_{i>1}m_i^2=0 \quad \text{or} \quad \sum_{i>1}m_i^2=3\ ,$$

and this shows that e_1 is the only halfperiod at which C is singular.

This completes the proof of the theorem. □

Remark 6.3. The statement about the numerical equivalence class of C in part (b) of the theorem leaves two possibilities: either $C\equiv k_0L$ or $C\equiv 2k_0L$. Let us stress here that both cases actually occur: If $2d+1$ is a square, then $(k_0,\ell_0)=(1,\sqrt{2d+1})$ is the minimal solution of Pell's equation, and the proof of [4, Theorem A.1(c)] shows that $C\equiv k_0L$ in this case. On the other hand, for $d=1$ we have $C\equiv 2k_0L$; this follows from the fact that $(k_0,\ell_0)=(2,3)$ and that the image of the unique divisor $\Theta\in|L|$ under the multiplication map $X\longrightarrow X$, $x\longmapsto 2x$, is an *irreducible* curve in $|4L|$ with multiplicity 6 at the origin (cf. [33]).

The theorem implies in particular that $\varepsilon(X,L)$ can be arbitrarily close to $\sqrt{L^2}$. Furthermore, it implies that the degree of the curve computing $\varepsilon(X,L)$ can be arbitrarily large, and that it cannot be bounded in terms of L^2 only:

Corollary 6.4. *For every real number $\delta>0$ and every integer $N>0$ there is an integer $d>0$ such that for every polarized abelian surface (X,L) of type $(1,d)$ with $NS(X)\cong\mathbb{Z}$ the following conditions hold:*

(a) $\sqrt{L^2}-\varepsilon(X,L)<\delta$.

(b) *The unique irreducible curve $C\subset X$ that computes $\varepsilon(X,L)$ at $x\in X$ satisfies the inequalities*

$$L\cdot C>N\cdot L^2 \quad \textit{and} \quad \operatorname{mult}_x C>N\sqrt{L^2}\ .$$

Proof. This follows essentially from the fact that for suitable d the solutions of Pell's equation are arbitrarily large. Specifically, let $d\geq 1$ be an integer such that $\sqrt{2d}$ is irrational, and let p_n/q_n, $n\geq 0$, be the convergents of $\sqrt{2d}$. One knows that for every solution (k,ℓ) of Pell's equation $\ell^2-2dk^2=1$, the rational number ℓ/k is one of the

convergents p_n/q_n (see e.g. [16, Chapter 10]). The sequences (p_n) and (q_n) have the following properties:

$$p_{n+1} > p_n \ , q_{n+1} > q_n \ , p_0 = a_0, q_0 = 1 \ , p_1 = a_1 a_0 + 1 \ , q_1 = a_1 \ ,$$

where

$$a_0 = \left\lfloor \sqrt{2d} \right\rfloor \qquad \text{and} \qquad a_1 = \left\lfloor \frac{1}{\sqrt{2d} - a_0} \right\rfloor \ .$$

We certainly have $(\ell, k) \neq (p_0, q_0)$, so that for the minimal solution (k_0, ℓ_0) we have the lower bound

$$k_0 \geq q_1 = \left\lfloor \frac{1}{\sqrt{2d} - \left\lfloor \sqrt{2d} \right\rfloor} \right\rfloor \ .$$

Since

$$\liminf \left\{ \sqrt{2d} - \left\lfloor \sqrt{2d} \right\rfloor \,\middle|\, d \geq 1, \ \sqrt{2d} \text{ irrational} \right\} = 0 \ ,$$

we can then choose d in such a way that $k_0 > N$, and hence $\ell_0 > k_0\sqrt{2d} > N\sqrt{2d}$. This gives (b). It follows from a calculation that then, after possibly repeating the argument with a larger N, the inequalities in (a) are satisfied as well. $\square$

Remark 6.5. Even though the formula for $\varepsilon(X, L)$ in Theorem 6.1 involves the solutions of a diophantine equation, the values for $\varepsilon(X, L)$ can be effectively computed in terms of d, since the solutions of Pell's equation can be obtained via continued fractions. In order to illustrate the situation, we include here a table providing the explicit (rounded) values of k_0, ℓ_0, $\varepsilon(X, L)$ and $\sqrt{2d}$ for $1 \leq d \leq 30$ (see Table 1). We know from Theorem 6.1 that the curve C_0 computing $\varepsilon(X, L)$ is of L-degree $2dk_0$ or $4dk_0$ and has a point of multiplicity l_0 or $2l_0$ respectively. Notice in particular how close $\varepsilon(X, L)$ is to the theoretical upper bound in the cases $d = 23$ and $d = 29$. The curve C_0 has multiplicity 24335 or 48670 for $d = 23$ and 19603 or 39206 for $d = 29$. It does not come as a surprise then that it is hard to find C_0 geometrically.

7. The nef cone of an abelian surface

We start with some remarks on cones. Let V be a real vector space and let Λ be a lattice in V. A *cone* in V is a subset $C \subset V$ such that $\mathbb{R}^+ \cdot C \subset C$. It is convex if and only if $C + C \subset C$. A cone is called *polyhedral* if there are finitely many elements $v_1, \ldots, v_r \in V$ such that

$$C = \sum_{i=1}^{r} \mathbb{R}^+ \cdot v_i \ ,$$

d	k_0	ℓ_0	$\varepsilon(X,L)$	$\sqrt{2d}$
1	2	3	1.333333333	1.414213562
2	–	–	2.000000000	2.000000000
3	2	5	2.400000000	2.449489743
4	1	3	2.666666667	2.828427125
5	6	19	3.157894737	3.162277660
6	2	7	3.428571429	3.464101615
7	4	15	3.733333333	3.741657387
8	–	–	4.000000000	4.000000000
9	4	17	4.235294118	4.242640687
10	2	9	4.444444444	4.472135955
11	42	197	4.690355330	4.690415760
12	1	5	4.800000000	4.898979486
13	10	51	5.098039216	5.099019514
14	24	127	5.291338583	5.291502622
15	2	11	5.454545455	5.477225575

d	k_0	ℓ_0	$\varepsilon(X,L)$	$\sqrt{2d}$
16	3	17	5.647058824	5.656854249
17	6	35	5.828571429	5.830951895
18	–	–	6.000000000	6.000000000
19	6	37	6.162162162	6.164414003
20	3	19	6.315789474	6.324555320
21	2	13	6.461538462	6.480740698
22	30	199	6.633165829	6.633249581
23	3588	24335	6.782329977	6.782329983
24	1	7	6.857142857	6.928203230
25	14	99	7.070707071	7.071067812
26	90	649	7.211093991	7.211102551
27	66	485	7.348453608	7.348469228
28	2	15	7.466666667	7.483314774
29	2574	19603	7.615773096	7.615773106
30	4	31	7.741935484	7.745966692

Table 1: The Seshadri constants $\varepsilon(X,L)$ of abelian surfaces (X,L) of type $(1,d)$ for $1 \le d \le 30$ when $NS(X) \cong \mathbb{Z}$.

and it is said to be *rational polyhedral* if the generators $v_1,\dots,v_r$ can be chosen within Λ. The dual $C^* \subset V^*$ of a cone C is the cone

$$C^* = \{\, w \in V^* \mid \langle w,v\rangle \ge 0 \text{ for all } v \in C \,\} \ .$$

One has $C = C^{**}$ if and only if C is closed and convex. C is (rational) polyhedral if and only if C^* is. Furthermore, by Gordon's Lemma, C is rational polyhedral if and only if the semi-group $C \cap \Lambda$ is finitely generated. (See e.g. [30, Theorem 14.1 and §§19,20] for the elementary properties of cones mentioned in this paragraph.)

Consider now a smooth projective variety X. Via the intersection product, the real vector space

$$N_1(X) =_{\rm def} \{\, \text{1-cycles on } X \text{ modulo numerical equivalence} \,\} \otimes \mathbb{R}$$

is dual to the Néron-Severi vector space $NS_{\mathbb{R}}(X) = NS(X) \otimes \mathbb{R}$. As usual denote by $NE(X)$ the *cone of curves* on X, i.e. the convex cone in $N_1(X)$ generated by the effective 1-cycles. The dual cone of $NE(X)$ is the nef cone

$$\mathrm{Nef}(X) = \{\, \lambda \in NS_{\mathbb{R}}(X) \mid \lambda\xi \ge 0 \text{ for all } \xi \in NE(X) \,\} \ ,$$

and the dual of $\mathrm{Nef}(X)$ is in turn the *closed cone of curves* $\overline{NE}(X)$, i.e. the closure of $NE(X)$ in $N_1(X)$, so

$$NE(X)^{**} = \mathrm{Nef}(X)^* = \overline{NE}(X) \ .$$

By the Cone Theorem [25] one knows that $\overline{NE}(X)$, and hence $\mathrm{Nef}(X)$, is rational polyhedral whenever $c_1(X)$ is ample. If $c_1(X)$ is not ample, however, the structure of $\overline{NE}(X)$

can be quite hard to determine and it will in general depend in a subtle way on the geometry of X (cf. [11, §4]). A good example for this phenomenon is [17] where $\overline{NE}(X)$ is studied for $K3$ surfaces.

Let now L be an $\mathbb{R}$-line bundle, i.e. an element of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. It is *ample* if the conditions of the Nakai-Moishezon criterion hold for L. By [9] this is equivalent to requiring that (the numerical equivalence class of) L belong to the interior of the nef cone $\mathrm{Nef}(X)$. Since we can take the pull-back of an $\mathbb{R}$-line bundle by a morphism, there is no problem to extend the definition of Seshadri constants to $\mathbb{R}$-line bundles L:

$$\begin{aligned} \varepsilon(L,x) &= \sup\{\, \varepsilon > 0 \mid f^*L - \varepsilon E \,\} \\ &= \inf\left\{ \frac{L\cdot C}{\mathrm{mult}_x C} \,\middle|\, C \subset X \text{ irreducible curve} \right\} , \end{aligned}$$

where $f : \mathrm{Bl}_x(X) \longrightarrow X$ denotes the blow-up at x and $E = f^{-1}(x)$. Again one has $\varepsilon(L,x) > 0$ for all $x \in X$, if L is ample.

As we will see, knowledge on the structure of the nef cone can be useful in the computation of Seshadri constants. Suppose for instance that $\mathrm{Nef}(X)$ is polyhedral (or, equivalently, that $\overline{NE}(X)$ is polyhedral), i.e.

$$\mathrm{Nef}(X) = \sum_{i=1}^{r} \mathbb{R}_0^+ \cdot [N_i] \tag{7.0.1}$$

with $\mathbb{R}$-line bundles N_i on X, so that if $L \in \mathrm{Pic}(X)$ is any ample line bundle, we have

$$L \equiv \sum_{i=1}^{r} a_i N_i$$

with real numbers $a_i > 0$. The Seshadri constant of L at a point $x \in X$ is then clearly bounded below in terms of the numbers a_i and the Seshadri constant of the line bundle $\sum_{i=1}^r N_i$:

$$\varepsilon(L,x) \geq \min\{a_1,\ldots,a_r\} \cdot \varepsilon\left(\sum_{i=1}^{r} N_i, x\right) . \tag{7.0.2}$$

Note that $\sum_{i=1}^r N_i$ is ample, so that the bound is indeed non-trivial (and in many cases even sharp, as we will see below).

Example 7.1. Consider a principally polarized abelian surface (X, L_0) with endomorphism ring $\mathrm{End}(X) \cong \mathbb{Z}[\sqrt{d}]$, where d is a square-free positive integer. Application of Shimura's theory shows that there is a two-dimensional family of such surfaces for any such d (see [6]). We know that $\varepsilon(L_0) = \frac{4}{3}$ (see [33] or Theorem 6.1). What can we say about the Seshadri constants of the other ample line bundles on X? First recall that the principal polarization induces an isomorphism of groups

$$NS(X) \longrightarrow \mathrm{End}^s(X),\ L \longmapsto \varphi_{L_0}^{-1} \circ \varphi_L ,$$

where $\mathrm{End}^s(X) \subset \mathrm{End}(X)$ is the subgroup of endomorphisms that are symmetric with respect to the Rosati involution $f \longmapsto \varphi_{L_0}^{-1} \circ f \circ \varphi_{L_0}$ on $\mathrm{End}(X)$. The endomorphism $\sqrt{d}$ has the characteristic polynomial $t^2 - d$, hence the corresponding line bundle $L_{\sqrt{d}} \in NS(X)$ satisfies $L_{\sqrt{d}}^2 = -2d$ and the classes of L_0 and $L_{\sqrt{d}}$ yield an orthogonal (with respect to the intersection form) basis of $NS(X)$. By the version of the Nakai-Moishezon Criterion given in [20, Corollary 4.3.3], a line bundle

$$aL_0 + bL_{\sqrt{d}} \quad \text{with } a, b \in \mathbb{Z}$$

is ample if and only if

$$(aL_0 + bL_{\sqrt{d}})^2 > 0 \quad \text{and} \quad (aL_0 + bL_{\sqrt{d}})L_0 > 0 \ .$$

This implies that the nef cone is given by

$$\begin{aligned} \mathrm{Nef}(X) &= \left\{ xL_0 + yL_{\sqrt{d}} \in \mathbb{R}^2 \,\middle|\, x \geq \sqrt{d}|y| \right\} \\ &= \mathbb{R}_0^+(\sqrt{d}L_0 + L_{\sqrt{d}}) + \mathbb{R}_0^+(\sqrt{d}L_0 - L_{\sqrt{d}}) \ . \end{aligned}$$

Note that it is polyhedral, but not rational polyhedral. For the generators $N^{\pm} = \sqrt{d}L_0 \pm L_{\sqrt{d}}$ we have

$$\varepsilon(N^+ + N^-) = \varepsilon(2\sqrt{d}L_0) = 2\sqrt{d} \cdot \varepsilon(L_0) = \frac{8}{3}\sqrt{d} \ .$$

so that (7.0.2) gives us the lower bound

$$\varepsilon(aN^+ + bN^-) \geq \frac{8}{3}\sqrt{d} \cdot \min\{a, b\} \ .$$

This bound is indeed sharp, as one sees by taking $a = b = \frac{1}{2\sqrt{d}}$, where one gets $aN^+ + bN^- = L_0$.

The previous example shows that knowledge about the nef cone of a variety can be useful for the computation of Seshadri constants. Consider now the case when X is an abelian variety. The most pleasant case, of course, is that $\mathrm{Nef}(X)$ is rational polyhedral, i.e. the case where the generators N_i in (7.0.1) can be taken as (integral) line bundles in $\mathrm{Pic}(X)$. By [2], this happens if and only if X is isogenous to a product

$$X_1 \times \ldots \times X_r$$

of mutually non-isogenous abelian varieties X_i of Picard number one. In general, the structure of $\mathrm{Nef}(X)$ can be more complicated. For the surface case, the following theorem gives the complete classification. We think that such a list is interesting, quite apart from its potential application on Seshadri constants. In the formulation of the proposition we distinguish the cases according to the rank of $NS(X)$. Recall that $1 \leq \mathrm{rank}\, NS(X) \leq 4$ for every abelian surface X.

Theorem 7.2. *Let X be an abelian surface, and let $\rho(X) = \operatorname{rank} NS(X)$ denote its Picard number.*

(a) *If $\rho(X) = 1$, then $\operatorname{Nef}(X) \cong \mathbb{R}_0^+$.*

(b) *Suppose $\rho(X) = 2$ and let L, L' be line bundles whose classes generate $NS(X) \otimes \mathbb{Q}$, with L being ample. Consider the integer*

$$\delta(L, L') =_{\text{def}} (L \cdot L')^2 - (L^2)((L')^2) \ .$$

Then $\operatorname{Nef}(X)$ is polyhedral,

$$\operatorname{Nef}(X) \cong \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x \geq \sqrt{\delta(L, L')} \cdot |y| \right\} \ .$$

If $\delta(L, L')$ is a square, then $\operatorname{Nef}(X)$ is rational polyhedral. In this case X is isogenous to a product $E_1 \times E_2$ of non-isogenous elliptic curves E_i with $\operatorname{End}(E_i) = \mathbb{Z}$.

If $\delta(L, L')$ is not a square, then $\operatorname{Nef}(X)$ is irrational polyhedral. In this case X is simple and has real or complex multiplication.

(c) *Suppose $\rho(X) = 3$. Then $\operatorname{Nef}(X)$ is a cone over a circle :*

$$\operatorname{Nef}(X) \cong \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, z^2 \leq xy, \ x \geq 0, \ y \geq 0 \right\} \ .$$

Either X is isogenous to the self-product $E \times E$ of an elliptic curve E with $\operatorname{End}(E) \cong \mathbb{Z}$, or X is simple and has indefinite quaternion multiplication (i.e. $\operatorname{End}_{\mathbb{Q}}(X)$ is an indefinite quaternion algebra).

(d) *Suppose $\rho(X) = 4$. Then X is isogenous to the self-product $E \times E$ of an elliptic curve E with complex multiplication, and $\operatorname{Nef}(X)$ is a cone over a half-sphere:*

$$\operatorname{Nef}(X) \cong \left\{ (x, y, z, t) \in \mathbb{R}^4 \,\middle|\, z^2 + t^2 \leq xy, \ x \geq 0, \ y \geq 0 \right\} \ .$$

Proof. (a) is clear. As for (b), consider the line bundle

$$M =_{\text{def}} (L \cdot L')L - L^2 \cdot L' \in \operatorname{Pic}(X) \ .$$

We have $L \cdot M = 0$ and $M^2 = -\delta(L, L') \cdot L^2$. Note that $\delta(L, L') > 0$, i.e. $M^2 < 0$, by the Hodge index theorem. By assumption, every line bundle on X is algebraically equivalent to a bundle $aL + bM$ with suitable $a, b \in \mathbb{Q}$. Now, $aL + bM$ is ample if and only if

$$(aL + bM)^2 > 0 \quad \text{and} \quad (aL + bM)L > 0 \ ,$$

and these conditions are satisfied if and only if

$$a^2 > b^2 \cdot \delta(L, L') \ .$$

This implies the statement about the nef cone and shows that it is in any event polyhedral. The quadratic form

$$\psi : \mathbb{Q}^2 \longrightarrow \mathbb{Q}, \quad (a, b) \longmapsto (aL + bM)^2$$

represents zero (non-trivially) if and only if $\sqrt{\delta(L,L')}$ is rational, i.e. if and only if $\mathrm{Nef}(X)$ is *rational* polyhedral. Now, if ψ represents zero,

$$0 = \psi(a,b) = \left(a^2 - b^2 \cdot \delta(L,L')\right) L^2 ,$$

then $(aL+bM)L > 0$, if we choose $a > 0$. But this implies that the class $aL+bM$ is effective and is represented by a multiple of an elliptic curve. So X is isogenous to a product $E_1 \times E_2$ of elliptic curves in this case. The condition $\rho(X) = 2$ implies that we must have $\mathrm{End}(E_i) = \mathbb{Z}$ and that the E_i are non-isogenous. In the alternative case, i.e. when ψ does not represent zero, then X is simple, and the classification of endomorphisms algebras of simple abelian varieties (see [20, Chap. 5] or [26, Sect. 21]) shows that then X has either real or complex multiplication.

(c) Consider first the case when X is simple. One sees from the classification of endomorphism algebras that then the assumption $\rho(X) = 3$ implies that X has indefinite quaternion multiplication. There is an isomorphism of the algebra $\mathrm{End}_{\mathbb{R}}(X)$ with $M_2(\mathbb{R})$ under which the Rosati involution on $\mathrm{End}_{\mathbb{R}}(X)$ corresponds to matrix transposition. The ample classes in $NS_{\mathbb{R}}(X)$ correspond under the composed map

$$NS_{\mathbb{R}}(X) \xrightarrow{\sim} \mathrm{End}^s_{\mathbb{R}}(X) \xrightarrow{\sim} \mathrm{Sym}_2(\mathbb{R})$$

to the positive definite matrices. This implies the statement on the nef cone. Suppose now that X is non-simple. The condition $\rho(X) = 3$ implies that then X is isogenous to a product $E \times E$, where E is an elliptic curve with $\mathrm{End}(E) = \mathbb{Z}$. The nef cone of $E \times E$, and hence also the nef cone of X, can be described explicitly in terms of a suitable basis of $NS_{\mathbb{Q}}(E \times E)$. Taking $\Delta \subset E \times E$ to be the diagonal, the classes of $E_1 = E \times 0, E_2 = 0 \times E$ and $F = \Delta - E_1 - E_2$ generate $NS_{\mathbb{Q}}(E \times E)$. A line bundle

$$M(a_1,a_2,b) =_{\mathrm{def}} a_1E_1 + a_2E_2 + bF$$

is ample if and only if

$$M(a_1,a_2,b)^2 > 0 \quad \text{and} \quad M(a_1,a_2,b) \cdot E_i > 0 \quad \text{for } i = 1,2 .$$

But these conditions are equivalent to $b^2 < a_1a_2$ and $a_i > 0$ respectively, and this proves the assertion.

(d) The condition $\rho(X) = 4$ implies that X is isogenous to $E \times E$, where E is an elliptic curve with complex multiplication (see [32]).

Then

$$\mathrm{End}_{\mathbb{Q}}(X) = M_2(\mathrm{End}_{\mathbb{Q}}(E)) = M_2(\mathbb{Q}(\sqrt{d})) ,$$

with an integer $d < 0$. Taking the Rosati involution with respect to a product polarization, we have

$$\mathrm{End}^s(X) = \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \;\middle|\; f_1, f_4 \in \mathrm{End}^s(E) \text{ and } f_2' = f_3 \right\}$$

The ample classes in $NS_{\mathbb{R}}(X)$ then correspond to the positive definite matrices of the form

$$\begin{pmatrix} \alpha & \beta+\gamma\sqrt{d} \\ \beta-\gamma\sqrt{d} & \eta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \eta \in \mathbb{R}$, and from this follows the assertion on the nef cone of X. □

8. Multiple point Seshadri constants on abelian surfaces

Consider a smooth projective variety X and an ample line bundle L on X. So far we have considered the Seshadri constant $\varepsilon(L,x)$ of L at a single point x of X. There is a natural generalization of this idea, a *multiple point Seshadri constant*, accounting for the positivity of L along a finite set of points. It is quite obvious what the natural definition is: For distinct points $x_1,\ldots,x_k$ in X one puts

$$\varepsilon(L,x_1,\ldots,x_k) =_{\text{def}} \sup\left\{ \varepsilon\in\mathbb{R} \,\middle|\, f^*L-\varepsilon\sum_{i=1}^k E_i \text{ nef} \right\}$$

where $f: Y \longrightarrow X$ is the blow-up of X at $x_1,\ldots,x_k$ and $E_i = f^{-1}(x_i)$ for $1 \le i \le k$ (cf. [21, (5.16)]). As in the one-point case, one has the equivalent definition

$$\varepsilon(L,x_1,\ldots,x_k) = \inf_C \frac{L\cdot C}{\sum_{i=1}^k \operatorname{mult}_{x_i} C} \ ,$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through at least one of the points $x_1,\ldots,x_k$.

There are the upper and lower bounds

$$\frac{1}{k}\min_{1\le i\le k}\varepsilon(L,x_i) \le \varepsilon(L,x_1,\ldots,x_k) \le \sqrt[n]{\frac{L^n}{k}} \ , \tag{8.0.1}$$

where $n=\dim X$. In fact, letting $\varepsilon=\varepsilon(L,x_1,\ldots,x_k)$, we have

$$L^n - k\varepsilon^n = \left(f^*L-\varepsilon\sum_{i=1}^k E_i\right)^n \ge 0 \ ,$$

since $f^*L-\varepsilon\sum E_i$ is nef, and this implies the second inequality in (8.0.1). Further, putting $\delta=\min\{\,\varepsilon(L,x_i)\,|\,1\le i\le k\,\}$, the line bundle

$$k\cdot f^*L-\delta\sum_{i=1}^k E_i = \sum_{i=1}^k (f^*L-\delta E_i)$$

is nef, and hence $\varepsilon(L,x_1,\ldots,x_k)\ge\delta/k$, which gives the first inequality in (8.0.1).

While already one-point Seshadri constants are subtle invariants, multiple point Seshadri constants are extremely hard to control. Suffice it to say that Nagata's famous conjecture relating the degrees and multiplicities of curves in the projective plane can equivalently be formulated as a statement on multiple point Seshadri constants on $\mathbb{P}^2$:

Nagata's Conjecture 8.1 (cf. [27]). *For general points $x_1, \dots, x_k \in \mathbb{P}^2$ with $k \geq 9$ one has*

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1), x_1, \dots, x_k) = \frac{1}{\sqrt{k}} .$$

In other words, the conjecture says that the multiple point Seshadri constant of $\mathcal{O}_{\mathbb{P}^2}(1)$ at ≥ 9 general points should have its maximal possible value. This is known to be true whenever k is a square.

In light of these facts it seems hardly surprising that only few general results on multiple point Seshadri constants are available (e.g. [18], [34]). Let us consider here the case of abelian surfaces. By homogeneity, the number $\varepsilon(L, x_1, \dots, x_k)$ depends then only on the differences $x_i - x_1$, $2 \leq i \leq k$, and we have

$$\varepsilon(L, x_1, \dots, x_k) \geq \frac{1}{k}\varepsilon(L, x_1) = \frac{1}{k}\varepsilon(L) .$$

The following result shows that in general (i.e. if either (X, L) is general or if the points $x_1, \dots, x_k$ are general on X) one has the strict inequality whenever $k \geq 2$. In fact, we show that equality can only hold for trivial reasons:

Proposition 8.2. *Let X be an abelian surface and let L be an ample line bundle on X. Suppose that $x_1, \dots, x_k \in X$ are distinct points with $k \geq 2$. If*

$$\varepsilon(L, x_1, \dots, x_k) = \frac{1}{k}\varepsilon(L) ,$$

then X contains an elliptic curve E with the property

$$L \cdot E = k \cdot \varepsilon(L, x_1, \dots, x_k)$$

and all the points $x_1, \dots, x_k$ lie on E.

The proposition will be deduced from the following result, which provides a lower bound on the multiple point Seshadri constant in terms of L^2 and k.

Proposition 8.3. *Let X be an abelian surface and let L be an ample line bundle. Then for every choice of points $x_1, \dots, x_k \in X$ with $k \geq 1$ we are in one of the following two cases:*

(a)

$$\varepsilon(L,x_1,\ldots,x_k) \geq \begin{cases} \dfrac{\sqrt{2L^2}}{k} & \text{, } \textit{if } k \geq 4 \\ \dfrac{\sqrt{L^2}}{2\sqrt{2}}\sqrt{\dfrac{8-k}{k}} & \text{, } \textit{if } 1 \leq k \leq 3 \end{cases}$$

or

(b) *X contains an elliptic curve E such that*

$$\varepsilon(L,x_1,\ldots,x_k) = \frac{L\cdot E}{\#\{\, i \mid x_i \in E\,\}} \ .$$

As one might expect, the bound in (a) increases with L^2 and decreases with k. It differs from the theoretical upper bound by a factor of order $1/\sqrt{k}$. Note that the appearance of a case dealing with the potential influence of elliptic curves on the Seshadri constant is quite inevitable: Polarized abelian surfaces (A,L) can contain elliptic curves of any given L-degree, no matter how large L^2 is.

Proof of Proposition 8.3. We consider first the contribution of the non-elliptic curves to the Seshadri constant, i.e. the number

$$\varepsilon'(L,x_1,\ldots,x_k) =_{\mathrm{def}} \inf\left\{ \frac{L\cdot C}{\sum_{i=1}^k \operatorname{mult}_{x_i} C} \;\middle|\; C\subset X \text{ ample irreducible curve} \right\} \ .$$

Note that the self-intersection of the curves $C\subset X$ in question is bounded below by

$$C^2 \geq 2 + \sum_{i=1}^k \operatorname{mult}_{x_i} C(\operatorname{mult}_{x_i} C - 1) \ . \tag{8.3.1}$$

In fact, since X contains no rational curves and since all elliptic curves are smooth, we have $p_g(C)\geq 2$, and hence

$$\frac{1}{2}C^2 - 1 \geq p_a(C) - p_g(C) \geq \sum_{i=1}^k \binom{\operatorname{mult}_{x_i} C}{2} \ ,$$

which implies (8.3.1).

Now, since $\sum_{i=1}^k(\operatorname{mult}_{x_i} C)^2 \geq (1/k)(\sum_{i_1}^k \operatorname{mult}_{x_i} C)^2$, the inequality (8.3.1) gives a quadratic relation for the sum $\sum_{i=1}^k \operatorname{mult}_{x_i} C$,

$$\left(\sum_{i=1}^k \operatorname{mult}_{x_i} C\right)^2 - k\sum_{i=1}^k \operatorname{mult}_{x_i} C + k(2-C^2) \leq 0 \ ,$$

which tells us that

$$\sum_{i=1}^{k} \operatorname{mult}_{x_i} C \le \frac{k}{2} + \sqrt{k\left(\frac{k}{4} + C^2 - 2\right)} \,.$$

Using now the Hodge index theorem for the line bundles L and $\mathcal{O}_X(C)$, this bound on the multiplicities yields a bound on the number $\varepsilon'(L, x_1, \ldots, x_k)$,

$$\varepsilon'(L, x_1, \ldots, x_k) \ge \inf_C \frac{L \cdot C}{\frac{k}{2} + \sqrt{k\left(\frac{k}{4} + \frac{(L\cdot C)^2}{L^2} - 2\right)}} \,,$$

where the infimum is taken over the ample irreducible curves $C \subset X$. Consider now for fixed numbers k and L^2 the real-valued function

$$f : t \longmapsto \frac{t}{\frac{k}{2} + \sqrt{k(\frac{k}{4} + \frac{t^2}{L^2} - 2)}} \,.$$

For $k < 8$ its minimum lies at the point

$$t_0 = \sqrt{2L^2}\sqrt{\frac{8-k}{k}} \,,$$

whereas for $k \ge 8$ the function is increasing. Note next that

$$t = L \cdot C \ge \sqrt{L^2}\sqrt{C^2} \ge \sqrt{2L^2} =_{\mathrm{def}} t_1$$

and that $t_0 \le t_1$ for $k \ge 4$. We conclude that

$$\varepsilon'(L, x_1, \ldots, x_k) \ge \min f|[t_1, \infty) = f(t_1) = \frac{\sqrt{2L^2}}{k}$$

for $k \ge 4$, and

$$\varepsilon'(L, x_1, \ldots, x_k) \ge f(t_0) = \frac{\sqrt{L^2}}{2\sqrt{2}}\sqrt{\frac{8-k}{k}}$$

for $k \le 3$.

Finally, if $\varepsilon(L, x_1, \ldots, x_k) < \varepsilon'(L, x_1, \ldots, x_k)$, then the multiple point Seshadri constant must be computed by an elliptic curve E, and one has $\varepsilon(L, x_1, \ldots, x_k) = L \cdot E/(\#\{\, i \mid x_i \in E\,\})$. $\square$

We now give the

Proof of Proposition 8.2. We will first show that under the hypothesis of the proposition the Seshadri constant $\varepsilon(L, x_1, \ldots, x_k)$ must be computed by an elliptic curve. Suppose to

the contrary that $\varepsilon(L,x_1,\ldots,x_k)=\varepsilon'(L,x_1,\ldots,x_k)$ (in the notation of the proof of the previous proposition). Then, using Proposition 8.3 and the hypothesis we get

$$\varepsilon(L,x_1)=k\cdot\varepsilon(L,x_1,\ldots,x_k)\geq\begin{cases}\sqrt{2L^2} & \text{, if } k\geq 4\\ \frac{1}{2\sqrt{2}}\sqrt{k(8-k)L^2} & \text{, if } 1\leq k\leq 3\end{cases}$$

On the other hand, one always has $\varepsilon(L,x_1)\leq\sqrt{L^2}$, so that

$$\frac{1}{k}\sqrt{L^2}\geq\begin{cases}\frac{\sqrt{2L^2}}{k} & \text{, if } k\geq 4\\ \frac{\sqrt{L^2}}{2\sqrt{2}}\sqrt{\frac{8-k}{k}} & \text{, if } 1\leq k\leq 3\end{cases}$$

In case $k\geq 4$ we certainly have a contradiction, and for $1\leq k\leq 3$ we get $(1/k)\sqrt{2d}\geq(1/2)\sqrt{2d(8-k)/(2k)}$, which is impossible due to the assumption $k\geq 2$.

So we conclude that $\varepsilon(L,x_1,\ldots,x_k)$ is computed by an elliptic curve E, and it remains to show that all the points $x_1,\ldots,x_k$ lie on E. But, putting $\ell=\#\{\,i\mid x_i\in E\,\}$, we find

$$\frac{L\cdot E}{\ell}=\varepsilon(L,x_1,\ldots,x_k)=\frac{1}{k}\varepsilon(L,x_1)\leq\frac{L\cdot E}{k}\ ,$$

which implies $\ell=k$, and this completes the proof. □

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